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Peirce's Criterion.

Values of κ.

	$\mu = 3.$					$\mu = 4.$			
m	n				m	n			
	1	2	3	4		1	2	3	4
10	1·72	1·44	1·28	1·16	10	1·63	1·37	1·22	1·11
12	1·83	1·55	1·38	1·26	12	1·76	1·49	1·33	1·22
14	1·92	1·64	1·47	1·35	14	1·86	1·59	1·43	1·31
16	2·00	1·72	1·55	1·42	16	1·94	1·67	1·50	1·38
18	2·06	1·78	1·61	1·48	18	2·01	1·74	1·57	1·45
20	2·12	1·84	1·67	1·54	20	2·07	1·80	1·63	1·51
25	2·23	1·96	1·79	1·66	25	2·19	1·92	1·76	1·63
30	2·32	2·05	1·88	1·75	30	2·29	2·02	1·85	1·73
35	2·39	2·12	1·95	1·83	35	2·36	2·10	1·93	1·81
40	2·45	2·19	2·02	1·90	40	2·43	2·16	2·00	1·88
45	2·51	2·24	2·08	1·95	45	2·48	2·22	2·06	1·94
50	2·55	2·29	2·13	2·00	50	2·53	2·27	2·11	1·99
60	2·63	2·37	2·21	2·09	60	2·61	2·35	2·19	2·08
70	2·69	2·44	2·28	2·16	70	2·67	2·42	2·26	2·15
80	2·74	2·49	2·34	2·22	80	2·73	2·48	2·32	2·21
90	2·79	2·54	2·39	2·27	90	2·77	2·53	2·37	2·26
100	2·83	2·58	2·43	2·32	100	2·81	2·57	2·42	2·31

On Oscillating Satellites. By H. C. Plummer, M.A.

1. In his paper "On Periodic Orbits"[*] (to be quoted here as *Ch.*) Professor Charlier has tried to obtain in a simple analytical manner some of the results already found by Professor Darwin in his well-known memoir[†] on this subject (to be quoted as *D.*). The orbits investigated by Professor Charlier are those of oscillating satellites—*i.e.* those small periodic orbits which can be described in the neighbourhood of the five points where exact solutions of the problem of three bodies are known to exist. Some of his results I have tried to discuss in a more general way in a paper[‡] (to be quoted as *Pl.*) in which I have considered the motion of a small body in the neighbourhood of centres of libration, or points of zero force, in any rotating field of force. In the present paper it is sought to push the analysis beyond the

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[*] *Meddelanden från Lunds Astronomiska Observatorium*, No. 18.

[†] *Acta Mathematica*, t. xxi. p. 99.

[‡] *Monthly Notices*, vol. lxii. p. 6.

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first degree of approximation previously obtained and to consider more particularly the orbits in the neighbourhood of the three collinear points of libration, especially those of families *a* and *b*, for which numerical results are to be found in *D.*

2. As far as possible the notation of *Pl.* will be retained. The equations of motion of the third mass in the most restricted form of the problem of three bodies, referred to axes rotating with the line joining the finite masses, are

$$\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y}$$

where

$$2\Omega = \mu\left(r_1^2 + \frac{2}{r_1}\right) + \nu\left(r_2^2 + \frac{2}{r_2}\right)$$

The masses of the finite bodies are μ and ν, and the unit of mass is such that $\mu + \nu = 1$. The origin is taken at the centre of gravity of the two revolving bodies, these lying on the axis of x and their distance apart being the unit of length. The unit of time is such that the angular velocity of the axes is also unity. The partial differentials of Ω with respect to x and y are: of the first order, p and q; of the second order, r, s, and t*; of the third order, P, Q, R, and S. Then if the point (x, y) is a centre of libration, for which $p = q = 0$, and the coordinates of the small mass are $x + \xi$, $y + \eta$, the equations of motion may be written

$$\ddot{\xi} - 2\dot{\eta} = r\xi + s\eta + \tfrac{1}{2}(\mathrm{P}\xi^2 + 2\mathrm{Q}\xi\eta + \mathrm{R}\eta^2) + \ldots$$
$$\ddot{\eta} + 2\dot{\xi} = s\xi + t\eta + \tfrac{1}{2}(\mathrm{Q}\xi^2 + 2\mathrm{R}\xi\eta + \mathrm{S}\eta^2) + \ldots$$

so long as the body continues within the region where the expansions are legitimate. To the first order of the relative coordinates the solution is already known. It is

$$\xi = h\cos(mt - \alpha), \quad \eta = k\cos(mt - \beta)$$

where

$$m^4 + (r + t - 4)m^2 + rt - s^2 = 0$$
$$h^2 = a^2(t + m^2), \quad k^2 = a^2(r + m^2)$$
$$\tan(\alpha - \beta) = -2m/s$$

These equations, found in *Pl.*, §§ 4 and 5†, leave the origin of time, depending on α, and the scale of the orbit, depending on a, arbitrary.

3. To proceed to the second approximation we may put

$$\xi = h\cos(mt - \alpha) + \xi', \qquad \eta = k\cos(mt - \beta) + \eta'$$

---

* The symbol t also represents the time; but this can hardly cause any confusion.

† The constant a used above is connected with the constant c used in *Pl.* by the relation $a^2(4m^2 + s^2) = c^2$.

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where h, k are small quantities of the first order, ξ', η' of the second order. In virtue of the preceding relations among the constants, the equations of motion now become

$$\left.\begin{aligned}\ddot{\xi}'-2\dot{\eta}'&=r\xi'+s\eta'+\mathrm{U}\\ \ddot{\eta}'+2\dot{\xi}'&=s\xi'+t\eta'+\mathrm{V}\end{aligned}\right\}\quad(\mathrm{A})$$

where if terms beyond the second order be neglected

$$2\mathrm{U}=\mathrm{P}h^2\cos^2(mt-\alpha)+2\mathrm{Q}hk\cos(mt-\alpha)\cos(mt-\beta)+\mathrm{R}k^2\cos^2(mt-\beta)$$

$$2\mathrm{V}=\mathrm{Q}h^2\cos^2(mt-\alpha)+2\mathrm{R}hk\cos(mt-\alpha)\cos(mt-\beta)+\mathrm{S}k^2\cos^2(mt-\beta)$$

Hence

$$\mathrm{U}=a_1+b_1\cos 2mt+c_1\sin 2mt$$
$$\mathrm{V}=a_2+b_2\cos 2mt+c_2\sin 2mt$$

where

$$4a_1=\mathrm{P}h^2+2\mathrm{Q}hk\cos(\alpha-\beta)+\mathrm{R}k^2$$
$$4b_1=\mathrm{P}h^2\cos 2\alpha+2\mathrm{Q}hk\cos(\alpha+\beta)+\mathrm{R}k^2\cos 2\beta$$
$$4c_1=\mathrm{P}h^2\sin 2\alpha+2\mathrm{Q}hk\sin(\alpha+\beta)+\mathrm{R}k^2\sin 2\beta$$

and the corresponding equations for a_2, b_2, and c_2 are found by substituting Q, R, S for P, Q, R respectively. Then the equations (A) will be satisfied by the solution

$$\xi'=f_1+h_1\cos 2mt+k_1\sin 2mt$$
$$\eta'=f_2+h_2\cos 2mt+k_2\sin 2mt$$

provided that

$$rf_1+sf_2+a_1=0$$
$$sf_1+tf_2+a_2=0$$

and that

$$\begin{array}{lllll}(4m^2+r)h_1+sh_2 & & +4mk_2 & & +b_1=0\\ & -4mh_2 & +(4m^2+r)k_1+sk_2 & & +c_1=0\\ sh_1 & +(4m^2+t)h_2-4mk_1 & & & +b_2=0\\ 4mh_1 & & +sk_1 & +(4m^2+t)k_2 & +c_2=0\end{array}$$

From this latter set of four equations we obtain

$$\begin{array}{llll}\delta\,.\,h_1=-(4m^2+t)b_1+sb_2 & & & +4mc_2\\ \delta\,.\,h_2=\quad sb_1 & -(4m^2+r)b_2-4mc_1 & & \\ \delta\,.\,k_1= & -4mb_2 & -(4m^2+t)c_1+sc_2 & \\ \delta\,.\,k_2=\quad 4mb_1 & & +sc_1 & -(4m^2+r)c_2\end{array}$$

where

$$\delta=3(4m^4+s^2-rt)$$

4. From these results obtained by proceeding to the second order of the dimensions of the relative orbit we have two general properties which characterise the orbits of oscillating satellites:

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(1) The orbits of any family are tautochronous to this order of approximation.

(2) The mean position of the small body is displaced by an amount of the second order from the corresponding centre of libration.

In this way it should be possible to find, with little extra trouble, better approximations to the relative paths of oscillating satellites than the elliptic orbits discussed in *Ch.* The results given in *D.* for satellites of families *a* and *b* show that tautochronism ceases to hold at a comparatively small distance from a centre of libration. This fact restricts the application of (1) to very limited areas, and suggests that an independent study of the variation of the period in this class of orbit would be very valuable.

5. The formulæ are much simplified when the collinear centres of libration in the special problem are considered. In this case $y=0$, and we have for the partial differentials of Ω

$$p=\pm\mu\,.\left(r_1-\frac{1}{r_1^2}\right)\pm\nu\,.\left(r_2-\frac{1}{r_2^2}\right)=0$$

$$q=0$$

$$r=\mu\left(1+\frac{2}{r_1^3}\right)+\nu\left(1+\frac{2}{r_2^3}\right)$$

$$s=0$$

$$t=\mu\left(1-\frac{1}{r_1^3}\right)+\nu\left(1-\frac{1}{r_2^3}\right)$$

$$P=\pm\mu\,.\left(-\frac{6}{r_1^4}\right)\pm\nu\,.\left(-\frac{6}{r_2^4}\right)$$

$$Q=0$$

$$R=-\tfrac{1}{2}P$$

$$S=0$$

The rule for the ambiguous signs is this : For the orbits between the finite masses (family *a*) we take $+\mu$, $-\nu$; for the orbits beyond ν (family *b*) we take $+\mu$, $+\nu$; for the orbits beyond μ (family *c*) we take $-\mu$, $-\nu$. Now for comparison with *D.*, $\mu=\frac{10}{11}$, $\nu=\frac{1}{11}$; and taking the values of r_1 and r_2 from *D.*, pp. 108–111, we obtain :

	Family *a*.	Family *b*.	Family *c*.
r_1	0·71751	1·34700	0·94693
r_2	0·28249	0·34700	1·94693
r	13·9878	6·0955	3·1660
t	− 5·4939	− 1·5478	− 0·0830
P	+ 65·076	− 39·279	+ 6·822
R	− 32·538	+ 19·639	− 3·411
m	2·6082	1·6763	1·0706
m^2	6·8025	2·8099	1·1463

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6. Further simplifications are effected by making $\alpha = 0$, $\beta = \frac{1}{2}\pi$. Then

$$
\begin{aligned}
4a_1 &= \mathrm{P}h^2 + \mathrm{R}k^2 = \mathrm{P}(h^2 - \tfrac{1}{2}k^2) = 4a^2\alpha_1 \\
4b_1 &= \mathrm{P}h^2 - \mathrm{R}k^2 = \mathrm{P}(h^2 + \tfrac{1}{2}k^2) = 4a^2\delta \, . \, \alpha_2 \\
4c_2 &= 2\mathrm{R}hk = -\mathrm{P}hk = 4a^2\delta \, . \, \alpha_3 \\
c_1 &= a_2 = b_2 = 0
\end{aligned}
$$

and hence

$$
\begin{aligned}
f_1 &= -a^2\alpha_1/r \\
h_1 &= -(4m^2 + t)a^2\alpha_2 + 4ma^2\alpha_3 \\
k_2 &= 4ma^2\alpha_2 - (4m^2 + r)a^2\alpha_3 \\
f_2 &= h_2 = k_1 = 0
\end{aligned}
$$

The following numerical values result :—

	Family *a*.	Family *b*.	Family *c*.
log δ	2·89534	2·09008	1·21887
log α_1	2·16976$_n$	1·49597	0·27043$_n$
log α_2	9·38435	9·65902$_n$	9·52077
log α_3	9·03335	9·42739$_n$	9·34366
log f_1/a^2	1·02401	0·71096$_n$	9·76992
h_1/a^2	−4·1352	+2·6261	−0·5487
k_2/a^2	−1·9209	+1·5800	−0·2895
log h/a	0·05840$_n$	0·05055$_n$	0·01333$_n$
log k/a	0·65893	0·47482	0·31736

The coefficients h and k are taken of opposite sign in accordance with the equation (*Pl.* p. 9).

$$(r + m^2)h \sin(\alpha - \beta) = 2mk$$

7. Thus the equations of the three families of orbits are obtained. They are, for family *a* :

$$
\begin{aligned}
\xi &= +10{\cdot}5685\, a^2 - 1{\cdot}1439\, a \cos 2{\cdot}6082t - 4{\cdot}1352\, a^2 \cos 5{\cdot}2164t \\
\eta &= \qquad\qquad\quad +4{\cdot}5597\, a \sin 2{\cdot}6082t - 1{\cdot}9209\, a^2 \sin 5{\cdot}2164t \, ;
\end{aligned}
$$

for family *b* :

$$
\begin{aligned}
\xi &= -\ 5{\cdot}1400\, a^2 - 1{\cdot}1234\, a \cos 1{\cdot}6763t + 2{\cdot}6261\, a^2 \cos 3{\cdot}3526t \\
\eta &= \qquad\qquad\quad +2{\cdot}9841\, a \sin 1{\cdot}6763t + 1{\cdot}5800\, a^2 \sin 3{\cdot}3526t \, ;
\end{aligned}
$$

and for family *c* :

$$
\begin{aligned}
\xi &= +\ 0{\cdot}5887\, a^2 - 1{\cdot}0312\, a \cos 1{\cdot}0706t - 0{\cdot}5487\, a^2 \cos 2{\cdot}1412t \\
\eta &= \qquad\qquad\quad +2{\cdot}0766\, a \sin 1{\cdot}0706t - 0{\cdot}2895\, a^2 \sin 2{\cdot}1412t
\end{aligned}
$$

8. The next step is to connect the parameter a with the con-

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stant C of the Jacobian integral $V^2 = 2\Omega - C$. The equations representing the orbits being of the form

$$\xi = h' \cos mt \,.\, a + (f_1' + h_1' \cos 2mt) \,.\, a^2$$
$$\eta = k' \sin mt \,.\, a + \qquad k_2' \sin 2mt \,.\, a^2$$

we have at the time $t = 0$,

$$\xi_0 = h' \,.\, a + (f_1' + h_1')\, a^2 \,;\; \eta_0 = 0$$
$$\dot{\xi}_0 = 0 \,;\; \dot{\eta}_0 = mk' \,.\, a + 2mk_2' \,.\, a^2$$

Hence to the third order in a,

$$V^2 = m^2 k'^2 \,.\, a^2 + 4m^2 k' k_2' \,.\, a^3$$
$$2\Omega = C_0 + r\xi_0^2 + \tfrac{1}{3} P\xi_0^3$$
$$= C_0 + rh'^2 \,.\, a^2 + \left\{2rh'(f_1' + h_1') + \tfrac{1}{3} Ph'^3\right\} \,.\, a^3$$

where C_0 is the value of 2Ω at the point of libration. Therefore

$$C - C_0 = (rh'^2 - m^2 k'^2) a^2 + \left\{2rh'(f_1' + h_1') - 4m^2 k' k_2' + \tfrac{1}{3} Ph'^3\right\} a^3$$

Now the coefficient of a^3 herein vanishes, as can be proved without difficulty. A useful check on the accuracy of the numbers occurring in § 7 is thus obtained, and the last equation can therefore be written in the simple form

$$C_0 - C = (m^4 - rt)\, a^2$$

which is equivalent to a formula in *Pl.* (§ 7). The numbers for the three cases are these:

	(*a*)	(*b*)	(*c*)
C_0	3·65292	3·53418	3·17322
$m^4 - rt$	123·13	17·33	1·576

9. That the coefficient of a^3 in the expansion of $C - C_0$ must vanish is clear from the fact that its sign would be changed by taking the initial conditions corresponding to $mt = \pi$. And the verification is easily obtained from the general relations of § 3 as simplified in §§ 5 and 6. For we have

$$\frac{h'}{k'} = \frac{-2m}{r + m^2} = \frac{t + m^2}{-2m} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$4rf_1' + Ph'^2 + Rk'^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$
$$(4m^2 + r)\, h_1' + 4mk_2' + \tfrac{1}{4}(Ph'^2 - Rk'^2) = 0 \quad \ldots \quad (3)$$
$$4mh_1' + (4m^2 + t)\, k_2' + \tfrac{1}{2} Rh'k' = 0 \quad \ldots \quad \ldots \quad (4)$$

Let (3) be multiplied by m and (4) by $(m^2 + r)$. The difference, when account is taken of (1), becomes

$$3mrh_1' + 3\,(m^2 + r)\, m^2 k_2' - \tfrac{1}{4} m\,(Ph'^2 + 3Rk'^2) = 0$$

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which again becomes when multiplied by $2h'/3m$

$$2rh'h_1' - 4m^2k'k_2' - \tfrac{1}{6}h'\,(\mathrm{P}h'^2 + 3\mathrm{R}k'^2) = 0$$

Hence by (2)

$$2rh'(f_1' + h_1') - 4m^2k'k_2' + \tfrac{1}{3}\mathrm{P}h'^3 = 0$$

which is the result required.

10. Four examples of orbits of family *a* are given in *D.* (pp. 233–238), the values of 11 C being 40·0, 39·0, 38·5, and 38·0. These give the numbers

$$11\,(\mathrm{C}_0 - \mathrm{C}) = 0{\cdot}1821,\quad 1{\cdot}1821,\quad 1{\cdot}6821,\quad 2{\cdot}1821$$

$$\log a = 8{\cdot}06428,\ 8{\cdot}47046,\ 8{\cdot}54705,\ 8{\cdot}60356$$

Whence the following four sets of equations for the orbits result :

$$(\alpha)\quad \xi = +{\cdot}0014 - {\cdot}0133 \cos mt - {\cdot}0006 \cos 2mt$$
$$\eta = +{\cdot}0529 \sin mt - {\cdot}0003 \sin 2mt$$
$$(\beta)\quad \xi = +{\cdot}0092 - {\cdot}0338 \cos mt - {\cdot}0036 \cos 2mt$$
$$\eta = +{\cdot}1347 \sin mt - {\cdot}0017 \sin 2mt$$
$$(\gamma)\quad \xi = +{\cdot}0131 - {\cdot}0403 \cos mt - {\cdot}0051 \cos 2mt$$
$$\eta = +{\cdot}1607 \sin mt - {\cdot}0024 \sin 2mt$$
$$(\delta)\quad \xi = +{\cdot}0170 - {\cdot}0459 \cos mt - {\cdot}0067 \cos 2mt$$
$$\eta = +{\cdot}1830 \sin mt - {\cdot}0031 \sin 2mt$$

For comparison with Professor Darwin's calculations the following numbers may suffice. They give the points where the orbits cross the axis of x and the coordinates of the points for which $mt = \frac{1}{2}\pi$; for the latter the ordinates corresponding to the computed abscissæ are found by interpolation among the coordinates given in *D.*

$mt =$	0	π		$\frac{1}{2}\pi$	
	ξ_0	ξ_1	$\xi_0 + \xi_1$	ξ'	η'
(α)	−·0125	+·0141	+·0016	+·0020	+·0529
D.	−·0125	+·0147	+·0022	,,	+·0530
(β)	−·0282	+·0394	+·0112	+·0128	+·1347
D.	−·0304	+·0413	+·0109	,,	+·1395
(γ)	−·0323	+·0483	+·0160	+·0182	+·1607
D.	−·0361	+·0523	+·0162	,,	+·1691
(δ)	−·0356	+·0562	+·0206	+·0237	+·1830
D.	−·041	+·060	+·019	,,	+·197

In every case ξ' is slightly less than the abscissa corresponding to the maximum ordinate in *D.*, as should be the case.

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11. Two examples of orbits of family *b* are given in *D.* (pp. 239–242), the values of 11 C being 38·5 and 38·0. These give

$$11\,(C_0-C) = 0{\cdot}3760, \qquad 0{\cdot}8760$$
$$\log a = 8{\cdot}64750, \qquad 8{\cdot}83116$$

and the two corresponding pairs of equations for the orbits are

$$(\alpha)\ \xi = -{\cdot}0101 - {\cdot}0499 \cos mt + {\cdot}0052 \cos 2\,mt$$
$$\eta = +{\cdot}1325 \sin mt + {\cdot}0031 \sin 2\,mt$$
$$(\beta)\ \xi = -{\cdot}0236 - {\cdot}0762 \cos mt + {\cdot}0121 \cos 2\,mt$$
$$\eta = +{\cdot}2023 \sin mt + {\cdot}0073 \sin 2\,mt$$

The results of a comparison with *D.* are given in a form similar to that used in the preceding paragraph.

$mt=$	0	π		$\frac{1}{2}\pi$	
	ξ_0	ξ_1	$\xi_0+\xi_1$	ξ'	η'
(α)	−·0548	+·0450	−·0098	−·0153	+·1325
D.	−·0551	+·0445	−·0106	,,	+·1312
(β)	−·0877	+·0647	−·0230	−·0357	+·2023
D.	−·0875	+·0679	−·0196	,,	+·1987

In both cases ξ' is slightly greater than the abscissa corresponding to the maximum ordinate in *D.*, as the above equations require.

12. Oscillating satellites belonging to the family *c* have not been investigated by Professor Darwin, and there seems to be no means of judging the accuracy of the above results for this family. The comparisons contained in §§ 10 and 11 for satellites of the families *a* and *b* seem to show a fairly good agreement, especially, as was to be expected, in the case of the smaller orbits. For the larger orbits, particularly those of family *a*, the results are by no means so good; and it may be noted that Professor Darwin himself found the orbit (δ) "exceedingly troublesome." But by going to a second approximation it is clear that a better notion has been obtained both of the mean position and shape of the orbit than was possible with the elliptic paths investigated by Professor Charlier. The results found in *Ch.* (*e.g.* p. 25) are really better than could fairly have been anticipated owing to the fact, mentioned in § 8, that the coefficient of a^3 in the development of C vanishes. Further progress in the study of the particular problem considered in the present paper seems to depend, as noticed in § 4, on a knowledge of the variation of the period and the effect of that variation on the form of the orbit.

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1903 *June* 11.